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ON PRODUCT OF $\mathcal{P}$-FUNCTIONS

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Riassunto

Si prova la produttività della $\mathcal{P}$-funzione per le proprietà topologiche $\mathcal{P} - T_0, T_1$, *Hausdorff, regular* introdotte da Pasynkov in [P_2] a $\mathcal{P}$ =*Urysohn, almost regular e semiregular* introdotte dall'autore e Cammaroto in [CN].

Abstract

It is shown that the $\mathcal{P}$-functions for the topological properties $\mathcal{P} = T_0$, T_1, *Hausdorff, regular,* introduced by Pasynkov in [P_2] and $\mathcal{P}$ =*Urysohn, almost regular, semiregular*, introduced by author and Cammaroto in [CN], are productive.

1. Introduction.

During the last two decades the idea to investigate the mappings as objects more general than spaces become rather

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Keywords. $\mathcal{P}$-function, product function, T_0-function, T_1-function, Hausdorff function, Urysohn function, regular function, almost-regular function, semiregularization, f-semiregularization, f-semiregular.

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popular. First approaches to this matter are due to Russian school and particularly to B. A. Pasynkov [P_1, P_2].

The concept of $\mathcal{P}$-function, i.e. a continuous function that satisfies a topological property $\mathcal{P}$ was introduced by author and Cammaroto in [CN] to extend the corresponding properties of $\mathcal{P}$-spaces. For a topological property, we define a property $\mathcal{P}$ for a function such that every continuous function on a $\mathcal{P}$-space is always a $\mathcal{P}$-function.

Recently Cammaroto, Fedorchuk and Porter [CFP] have studied the $\mathcal{P}$-functions for the property $\mathcal{P} = H$*-closure*.

In this paper we will show that the $\mathcal{P}$-functions for the topological properties $\mathcal{P} = T_0$, T_1, *Hausdorff, regular* introduced by Pasynkov in [P_2] and $\mathcal{P} =$*Urysohn, semiregular, almost regular*, introduced by author and Cammaroto in [CN], are productive, i.e. that a product of functions has the property $\mathcal{P}$ iff each function has $\mathcal{P}$.

2. Preliminaries.

Throughout this paper, all hypothesized functions are assumed to be continuous unless it is stated otherwise.

For notations, definitions or basic properties not explicitly mentioned here we refer to [E] and [PW].

Let X be a topological space, in all the sequel, $\tau(X)$ will denote the set of open sets of X and $\sigma(X)$ will denote the set of closed sets of X. If Z is a subset of X, then $\tau(X)_{|Z}$ will denote the relative open topology on Z of $\tau(X)$ while $\sigma(X)_{|Z}$ will denote the relative closed topology on Z of X.

Let X be a topological space, a subset $A \subseteq X$ is said regular open if it is the interior of its own closure or, equivalently, if it is the interior of some closed set, while it is said regular closed if it is the closure of its own interior or, equivalently, if it is the closure of some open set. We denote by $RO(X)$ and $RC(X)$ respectively the set of regular open subsets of X and the set of regular closed subsets of X.

Let X be a topological space and $A, B, X' \subseteq X$ be subsets. Then A and B are said separated by neighbourhoods in X' if the sets $A \cap X'$ and $B \cap X'$ have disjoint neighbourhoods in the topological space X' relative to X, that is there are open sets $U, V \in \tau(X')$ such that $A \cap X' \subseteq U$, $B \cap X' \subseteq V$ and $U \cap V = \emptyset$.

A topological space X is said almost regular (see [SA]) if any regular closed set and any singleton disjoint from it can be separated by neighbourhoods in the space X.

It is known that the set $RO(X)$ forms an open base for a topology on X. The topological space on X equipped with the topology generated by $RO(X)$ is usually denoted by $X(s)$ and it is called the semiregularization of X.

A topological space X is said semiregular if the set $RO(X)$ of the regular open subsets of X forms an open base for X, i.e. if $X = X(s)$.

Let X and Y be two topological spaces, and $f \in C(X, Y)$ a function from X to Y, then we will say that:

- f is T_0 if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there is some neighbourhood U of x which doesn't contain y or some neighbourhood U' of y which doesn't contain x;
- f is T_1 if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there is some neighbourhood U of x which doesn't contain y;
- f is Hausdorff (T_2) if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there are two disjoint open sets containing

x and y;

- f is Urysohn ($T_{2\frac{1}{2}}$) [CN] if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there are some open neighbourhood O of $f(x)$ in Y and open subsets $U, V \in \tau(f^{\leftarrow}(O))$ such that $x \in U$, $y \in V$ and $cl_{f^{\leftarrow}(O)}(U) \cap cl_{f^{\leftarrow}(O)}(V) = \emptyset$;

- f is regular if for each closed set F and $x \in X \setminus F$ there is some open neighbourhood O of $f(x)$ in Y such that $\{x\}$ and F are separated by neighbourhoods in $f^{\leftarrow}(O)$.

- f is almost regular [CN] if for each $C \in RC(X)$ and $x \in X \setminus C$ there is an open neighbourhood O of $f(x)$ in Y such that $\{x\}$ and C are separated by neighbourhoods in $f^{\leftarrow}(O)$;

- f is semiregular [CN] if for each $A \in \tau(X)$ and $x \in A$ there are an open neighbourhood O of $f(x)$ in Y and a regular open subset $R \in RO(f^{\leftarrow}(O))$ such that $x \in R \subseteq A \cap f^{\leftarrow}(O)$.

It is immediate to check that each function defined on a T_0, T_1, Hausdorff, Urysohn, regular, almost regular or semiregular space is respectively a T_0, T_1 Hausdorff, Urysohn, regular, almost regular or semiregular function, i.e. that these are well definitions of $\mathcal{P}$-functions. It is easy to prove that every Hausdorff function is a T_1-function and that each T_1-function is a T_0-function. Moreover, in [CN] it is shown that every Urysohn function is Hausdorff, that every regular function is almost regular and semiregular and that, in general, all the converses are false. In that same paper, it is also proved that every almost regular, Hausdorff function is Urysohn and that a function is regular iff it is almost regular and semiregular.

The following characterizations of regular and semiregular functions are easy but useful.

PROPOSITION 2.1. *Let X and Y be two topological spaces. Then $f \in C(X, Y)$ is regular iff for each (basic) open set $A \in \tau(X)$ and $x \in A$ there are $O \in \mathcal{U}_{f(x)}$ and $U \in \tau(f^{\leftarrow}(O))$ such that $x \in U \subseteq cl_{f^{\leftarrow}(O)}(U) \subseteq Anf^{\rightarrow}(0)$.*

PROPOSITION 2.2. [CN]. *Let X and Y be two spaces. Then $f \in C(X, Y)$ is semiregular iff for each $x \in X$ and $V \in \mathcal{U}_x$ there are $O \in \mathcal{U}_{f(x)}$ and $U \in \tau(X)$ such that $x \in U \subseteq int_X(cl_X(U)) \cap f^{\leftarrow}(O) \subseteq V \cap f^{\leftarrow}(O)$.*

Notation 2.3. [CN]. Let X and Y be two spaces and $f \in C(X, Y)$ a function from X to Y, we consider the set $\mathcal{S} = \{int_X(cl_X(U)) \cap f^{\leftarrow}(O) : U \in \tau(X), \ O \in \tau(Y)\} = \{int_{f^{\leftarrow}(O)}(cl_{f^{\leftarrow}(O)}(W)) : W \in \tau(f^{\leftarrow}(O)), \ O \in \tau(Y)\}$. It is easy to verify that $\mathcal{S}$ forms an open base for a topology on X. We will denote with $X(s, f)$ the topological space on X equipped with the topology generated by $\mathcal{S}$.

The following proposition it is easy to verify.

PROPOSITION 2.4. [CN]. *Let X be a space, $U \in \tau(X)$ and $F \in \sigma(X)$. Then:*

(1) if X is Hausdorff, so is $X(s, f)$.

(2) $\tau(X(s)) \subseteq \tau(X(s, f)) \subseteq \tau(X)$.

(3) $cl_X(U) = cl_{X(s,f)}(U)$, $int_X(F) = int_{X(s,f)}(F)$.

(4) $int_X(cl_X(U)) = int_{X(s,f)}(cl_{X(s,f)}(U))$,
$cl_X(int_X(F)) = cl_{X(s,f)}(int_{X(s,f)}(F))$.

(5) $RO(X) = RO(X(s, f))$, $RC(X) = RC(X(s, f))$.

(6) $(X(s, f))(s, f) = X(s, f)$.

Then the space $X(s, f)$ verifies the same properties of the semiregularization $X(s)$ of X (see, for example, [PW]) and for this reason it is natural to call it the *f-semiregularization* of X. We say also that X is *f-semiregular* if $X = X(s, f)$.

Notation 2.5. [CN]. Let X and Y be two spaces and $f \in C(X, Y)$ a function between them, we denote by $f(s)$ the function $f : X(s, f) \to Y$. By definition of $\mathcal{S}$, it is clear that $f(s) \in (C(X(s, f), Y)$.

In the sequel will be useful the following proposition.

PROPOSITION 2.6. [CN]. *Let $f \in C(X, Y)$ a function from X to Y, then f is semiregular iff X is f-semiregular.*

Remark 2.7. Clearly $f = f(s)$ iff $X = X(s, f)$ and so, we can say that f is semiregular iff $f = f(s)$. Hence it is obvious that $f(s)$ is semiregular.

Moreover, we have this important:

PROPOSITION 2.8. [CN]. *Let $f \in C(X, Y)$ a function from X to Y, then f is almost regular iff $f(s)$ is regular.*

Let X and Y be two topological spaces, $f \in C(X, Y)$ a function from X to Y and X' a subset of X, a restriction $f_{|X'} \in C(X', Y)$ of f to X' is said open (dense) if its domain X' is an open (dense) subset of X.

We will say that a property $\mathcal{P}$ for function is (open; dense) hereditary iff every (open; dense) restriction of a $\mathcal{P}$-function is again a $\mathcal{P}$-function.

PROPOSITION 2.9. [CN].

(1) The $T_0, T_1, T_2, T_{2\frac{1}{2}}$ properties for functions are hereditary.

(2) The regularity for functions is hereditary.

(3) The semiregularity for functions is open (dense) hereditary.

(4) The almost-regularity for functions is open (dense) hereditary.

3. Questions of productivity.

Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two sets of spaces, $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$ their product spaces and let $\{f_i\}_{i\in I}$ be a set of function $f_i \in F(X_i, Y_i)$. We recall that the product function $f = \prod_{i\in I} f_i$ is the member of $F(X,Y)$ defined by $\Pi_i^Y \circ f = f_i \circ \Pi_i^X$ for each $i \in I$ (where $\Pi_i^X : X \to X_i$ and $\Pi_i^Y : Y \to Y_i$ are respectively the i-th projection functions of X onto X_i and Y onto Y_i).

It is easy to prove (see [PW]) that if every f_i is a continuous function, then so is the product function $f = \prod_{i\in I} f_i$. Thus, we can consider the product function as $\mathcal{P}$-function.

THEOREM 3.1. *The product function $f = \prod_{i\in I} f_i$ is T_0 iff each function f_i is T_0.*

Proof. ($\Rightarrow$) Let $f \in C(X,Y)$ be T_0. Fixed $k \in I$, we consider the function $f_k \in C(X_k, Y_k)$. Let $x, y \in X_k$ such that $x \neq y$ and $f_k(x) = f_k(y)$. Then, picked an arbitrary point $\xi \in X$, we consider the slice $S = S(\xi, k)$ of X through ξ and parallel to X_k. As, from 2.9(1) the T_0 property for functions is hereditary, $f_{|S} \in C(S,Y)$ is T_0. Then, the points $p = \langle p_i \rangle_{i\in I}$, $q = \langle q_i \rangle_{i\in I} \in S$, defined by $p_k = x$ and $q_k = y$, are such that $p \neq q$ and $f_{|S}(p) = f_{|S}(q)$. Thus, there is a basic open neighbourhood U of p in S which doesn't contain q or a basic neighbourhood U' of q in S which doesn't contain p. Supposed, for example, that exists a basic open neighbourhood

$U = \prod_{i \in I} U_i$ of p in S, then $U_k \in \tau(X_k)\setminus\{X_k\}$ is an open neighbourhood of x in X_k which doesn't contain y. Since the other case is perfectly analogous, this proves that each function $f_k \in C(X_k, Y_k)$ is T_0.

($\Leftarrow$) Let $x = \langle x_i \rangle_{i \in I}$, $y = \langle y_i \rangle_{i \in I}$ such that $x \neq y$ and $f(x) = f(y)$, i.e. such that $f_i(x_i) = f_i(y_i)$ for each $i \in I$ and there is $j \in I$ such that $x_j \neq y_j$. As, in particular, $f_j \in C(X_j, Y_j)$ is T_0, there is an open neighbourhood U_j of x_j in X_j which doesn't contain y_j, or an open neighbourhood U_j of y_j in X_j which doesn't contain x_j. Then, assumed $U = \prod_{i \in I} U_i$ with $U_i = X_i$ for each $i \in I\setminus\{j\}$, U is a basic open neighbourhood of x in X not containing y or a basic open neighbourhood of y in X not containing x. Thus the product function $f \in C(X, Y)$ is T_0. ■

THEOREM 3.2. *The product function* $f = \prod_{i \in I} f_i$ *is* T_1 *(Hausdorff) iff each function* f_i *is* T_1 *(Hausdorff).*

Proof. Similar to the proof of 3.1. ■

THEOREM 3.3. *The product function* $f = \prod_{i \in I} f_i$ *is Urysohn iff each function* f_i *is Urysohn.*

Proof. ($\Rightarrow$) Let $f \in C(X, Y)$ be Urysohn. Fixed $k \in I$, we consider the function $f_k \in C(X_k, Y_k)$. Let $x, y \in X_k$ such that $x \neq y$ and $f_k(x) = f_k(y)$. Then, picked an arbitrary point $\xi \in X$, we consider the slice $S = \prod_{i \in I} S_i = S(\xi, k)$ of X through ξ and parallel to X_k. As, from 2.9(1) the $T_{2\frac{1}{2}}$ property for functions is hereditary, $f_{|S} \in C(S, Y)$ is Urysohn. Then, the points $p = \langle p_i \rangle_{i \in I}$, $q = \langle q_i \rangle_{i \in I} \in S$, defined by $p_k = x$ and

$q_k = y$ are such that $p \neq q$ and $f_{|S}(p) = f_{|S}(q)$. Thus there are a basic open neighbourhood $O = \prod_{i \in I} O_i$ of $f_{|S}(p)$ in Y and two basic open sets $U = \prod_{i \in I} U_i$, $V = \prod_{i \in I} V_i \in \tau(f_{|S}^{\leftarrow}(O))$ such that $p \in U$, $q \in V$ and $cl_{f_{|S}^{\leftarrow}(O)}(U) \cap cl_{f_{|S}^{\leftarrow}(O)}(V) = \emptyset$. Then, $f_{|S}^{\leftarrow}(O) = f^{\leftarrow}(O) \cap S = \prod_{i \in I} f^{\leftarrow}(O_i) \cap \prod_{i \in I} S_i = \prod_{i \in I} (f_i^{\leftarrow}(O_i) \cap S_i) = \prod_{i \in I} f_{i|S_i}^{\leftarrow}(O_i)$, and so $\prod_{i \in I} cl_{f_{i|S_i}^{\leftarrow}(O_i)}(U_i) \cap \prod_{i \in I} cl_{f_{i|S_i}^{\leftarrow}(O_i)}(V_i) = \emptyset$. Since, obviously $S_k = X_k$ and $f_{k|S_k}^{\leftarrow}(O_k) = f_k^{\leftarrow}(O_k)$, we have that O_k is an open neighbourhood of $f_k(x)$ in Y_k and that U_k and V_k are two open neighbourhoods of x and y in $f_k^{\leftarrow}(O_k)$ such that $cl_{f_k^{\leftarrow}(O_k)}(U_k) \cap cl_{f_k^{\leftarrow}(O_k)}(V_k) = \emptyset$. Thus $f_k \in C(X_k, Y_k)$ is Urysohn.

($\Leftarrow$) Conversely, we suppose that each $f_i \in C(X_i, Y_i)$ is Urysohn. Let $x = \langle x_i \rangle_{i \in I}$, $y = \langle y_i \rangle_{i \in I} \in X$ such that $x \neq y$ and $f(x) = f(y)$, i.e. that $f_i(x_i) = f_i(y_i)$ for each $i \in I$ and there is $j \in I$ such that $x_j \neq y_j$. As, in particular, $f_j \in C(X_j, Y_j)$ is Urysohn, there are an open neighbourhood O_j of $f_j(x)$ in Y_j and two open sets U_j, $V_j \in \tau(f_j^{\leftarrow}(O_j)) = \tau(X_j)_{|f_j^{\leftarrow}(O_j)}$ such that $x \in U_j$, $y \in V_j$ and $cl_{f_j^{\leftarrow}(O_j)}(U_j) \cap cl_{f_j^{\leftarrow}(O_j)}(V_j) = \emptyset$. Then, assumed $O = \prod_{i \in I} O_i$, $U = \prod_{i \in I} U_i$ and $V = \prod_{i \in I} V_i$ putting $O_i = Y_i$ and $U_i = V_i = X_i$ for each $i \in I \setminus \{j\}$, O is an open neighbourhood of $f(x)$ in Y while U and V are two open neighbourhoods of x and y in X. As $f^{\leftarrow}(O) = \prod_{i \in I} f_i^{\leftarrow}(O_i)$, we have that $cl_{f^{\leftarrow}(O)}(U) \cap cl_{f^{\leftarrow}(O)}(V) = \left(\prod_{i \in I} cl_{f_i^{\leftarrow}(O_i)}(U_i) \right) \cap \left(\prod_{i \in I} cl_{f_i^{\leftarrow}(O_i)}(V) \right) = \emptyset$. So, it is proved that the product function $f \in C(X, Y)$ is Urysohn. ■

THEOREM 3.4. *The product function* $f = \prod_{i \in I} f_i$ *is regular iff each function* f_i *is regular.*

Proof. ($\Rightarrow$) Let $f \in C(X, Y)$ be regular. Fixed $k \in I$, we consider the function $f_k \in C(X_k, Y_k)$. Let $F \in \sigma(X_k)$ and $x \in X_k \setminus F$. Then, picked an arbitrary point $\xi \in X$, we consider the slice $S = \prod_{i \in I} S_i = S(\xi, k)$ of X through ξ and parallel to X_k. As, from 2.9(2) the regularity for functions is hereditary, $f_{|S} \in C(S, Y)$ is regular. So, if we consider the point $p = \langle p_i \rangle_{i \in I} \in S$, defined by $p_k = x$ and the basic closed set $C = \prod_{i \in I} C_i$ of S, defined by $C_k = F$, it is clear that $p \in S \setminus C$. Thus there are a basic open neighbourhood $O = \prod_{i \in I} O_i$ of $f_{|S}(p)$ in Y and two basic open sets $U = \prod_{i \in I} U_i$, $V = \prod_{i \in I} V_i \in \tau(f_{|S}^{\leftarrow}(O))$ such that $p \in U$, $C \cap f_{|S}^{\leftarrow}(O) \subseteq V$ and $U \cap V = \emptyset$. As, $f_{|S}^{\leftarrow}(O) = \prod_{i \in I} f_{i|S_i}^{\leftarrow}(O_i)$ (see ($\Rightarrow$) of 3.3), $\prod_{i \in I}(C_i \cap f_{i|S_i}^{\leftarrow}(O_i)) = \prod_{i \in I} C_i \cap \prod_{i \in I} f_{i|S_i}^{\leftarrow}(O_i) = C \cap f_{|S}^{\leftarrow}(O) \subseteq V = \prod_{i \in I} V_i$.

Since, obviously $S_k = X_k$ and $f_{k|S_k}^{\leftarrow}(O_k) = f_k^{\leftarrow}(O_k)$ O_k is an open neighbourhood of $f_k(x)$ in Y_k while U_k and V_k are two open sets of $f_k^{\leftarrow}(O_k)$ such that $x \in U_k$, $F \cap f_k^{\leftarrow}(O_k) \subseteq V_k$ and $U_k \cap V_k = \emptyset$. This proves that $f_k \in C(X_k, Y_k)$ is regular.

($\Leftarrow$) Conversely, we suppose that each $f_i \in C(X_i, Y_j)$ is regular. Let $A = \prod_{i \in I} A_i \in \tau(X)$ a basic open set of X and $x = \langle x_i \rangle_{i \in I} \in A$. So, there exists a finite set $F \subseteq I$ such that $x_i \in A_i \in \tau(X_i) \setminus \{X_i\}$ for each $i \in F$. Since each $f_i \in C(X_i, Y_i)$ (with $i \in F$) is regular, by 2.1, there are a neighbourhood O_i of $f_i(x)$ in Y_i and an open set $U_i \in \tau(f_i^{\leftarrow}(O_i)) = \tau(X_i)_{|f_i^{\leftarrow}(O_i)}$ such that $x_i \in U_i \subseteq cl_{f_i^{\leftarrow}(O_i)}(U_i) \subseteq A_i \cap f_i^{\leftarrow}(O_i)$. Then, assumed $O = \prod_{i \in I} O_i$ and $U = \prod_{i \in I} U_i$ putting $O_i = Y_i$ and $U_i = X_i$ for each $i \in I \setminus F$, we have that O is a neighbourhood of

$f(x)$ in Y and that U is an open set of X such that $x \in U \subseteq cl_{f^{\leftarrow}(O)}(U) = \prod_{i\in I} cl_{f_i^{\leftarrow}(O_i)}(U_i) \subseteq \prod_{i\in I}(A_i \cap f_i^{\leftarrow}(O_i)) = \prod_{i\in I} A_i \cap \prod_{i\in I} f_i^{\leftarrow}(O_i) = A \cap f^{\leftarrow}(O)$. So, by 2.1, the product function $f \in C(X, Y)$ is regular. ■

To prove the productivity of $\mathcal{P}$-functions for the properties $\mathcal{P}$ =*almost regular and semiregular* we need the following:

LEMMA 3.5. $\prod_{i\in I} f_i(s) = \left(\prod_{i\in I} f_i\right)(s)$.

Proof. It suffices to prove that $\tau\left(\prod_{i\in I} X_i(s, f_i)\right) = \tau(X(s, f))$. Let $B = \prod_{i\in I} B_i$ a basic open subset of $\prod_{i\in I} X_i(s, f_i)$. Then there exist $F \subseteq I$ finite such that $B_i \in \tau(X_i(s, f_i))\backslash\{X_i\}$ for each $i \in F$ and $B_i = X_i$ for each $i \in I\backslash F$. Hence, there are U_i and O_i such that $B_i = int_{X_i}(cl_{X_i}(U_i)) \cap f_i^{\leftarrow}(O_i)$ with $U_i \in \tau(X_i)\backslash\{X_i\}$, $O_i \in \tau(Y_i)\backslash\{Y_i\}$ for each $i \in F$ and $U_i = X_i$, $O_i = Y_i$ for each $i \in I\backslash F$. So, $B = \prod_{i\in I} B_i = \prod_{i\in I}(int_{X_i}(cl_{X_i}(U_i)) \cap f_i^{\leftarrow}(O_i)) = \left(\prod_{i\in I} int_{X_i}(cl_{X_i}(U_i))\right) \cap \left(\prod_{i\in I} f_i^{\leftarrow}(O_i)\right) = int_X\left(cl_X\left(\prod_{i\in I} U_i\right)\right) \cap f^{\leftarrow}\left(\prod_{i\in I} O_i\right)$. Since, $\prod_{i\in I} U_i \in \tau(X)$ and $f(x) \in \prod_{i\in I} O_i \in \tau(Y)$, it is clear that $B \in \tau(X(s, f))$.

On the other hand, let $B' = int_X(cl_X(U)) \cap f^{\leftarrow}(O)$, with $U \in \tau(X)$ and $O \in \tau(Y)$, a basic open set of $X(s, f)$ and let $x \in B'$. Hence, $x \in int_X(cl_X(U))$ and $f(x) \in O$. Then, there are a basic open set $\prod_{i\in I} W_i \in \tau(X)$ such that

$x \in \prod_{i\in I} W_i \subseteq int_X(cl_X(U))$ and a basic open set $\prod_{i\in I} A_i \in \tau(Y)$ such that $f(x) \in \prod_{i\in I} A_i \subseteq O$. So, $x \in \prod_{i\in I} f_i^{\leftarrow}(A_i) = f^{\leftarrow}\left(\prod_{i\in I} A_i\right) \subseteq f^{\leftarrow}(O)$. Moreover, $\prod_{i\in I} int_{X_i}(cl_{X_i}(W_i)) = int_X\left(cl_X\left(\prod_{i\in I} W_i\right)\right) \subseteq int_X(cl_X(int_X(cl_X(U)))) = int_X(cl_X(U))$. So, $x \in \prod_{i\in I}(int_{X_i}(cl_{X_i}(W_i)) \cap f_i^{\leftarrow}(A_i)) = \prod_{i\in I} int_{X_i}(cl_{X_i}(Wi)) \cap \prod_{i\in I} f_i^{\leftarrow}(A_i) \subseteq int_X(cl_X(U)) \cap f^{\leftarrow}(O) = B'$, where $\prod_{i\in I}(int_{X_i}(cl_{X_i}$ $(W_i)) \cap f_i^{\leftarrow}(A_i))$ belongs to the base of $\tau\left(\prod_{i\in I} X_i(s, f_i)\right)$. Thus, the lemma is proved. ■

THEOREM 3.6. *The product function* $f = \prod_{i\in I} f_i$ *is almost regular iff each function* f_i *is almost regular.*

Proof. In fact, by 2.8, we have that $\prod_{i\in I} f_i$ is almost regular iff $\left(\prod_{i\in I} f_i\right)(s)$ is regular i.e., by 3.5, iff $\prod_{i\in I} f_i(s)$ is regular and, by 3.4, iff each $f_i(s)$ is regular, that is, by 2.8, iff each f_i is almost regular. ■

THEOREM 3.7. *The product function* $f = \prod_{i\in I} f_i$ *is semiregular iff each function* f_i *is semiregular.*

Proof. $(\Rightarrow)$ Let $f \in C(X, Y)$ semiregular. Fixed $k \in I$, we consider the function $f_k \in C(X_k, Y_k)$. Let $V \in \tau(X_k)$ and $x \in V$. Then, picked an arbitrary point $\xi_i \in X_i$ for each $i \in I\backslash\{k\}$, we consider the point $p = \langle p_i\rangle_{i\in I} \in X$, defined by $p_k = x$ and

$p_i = \xi_i$ for each $i \in I \setminus \{k\}$, and the basic open set $A = \prod_{i \in I} A_i$ of the product space X, defined by $A_k = V$ and $A_i = X_i$ for each $i \in I \setminus \{k\}$. It is clear that $p \in A$. So, by 2.2, there are a basic open neighbourhood $O = \prod_{i \in I} O_i$ of $f(p)$ in Y and a basic open set $U = \prod_{i \in I} U_i \in \tau(f^{\leftarrow}(O)) = \tau(X)_{|f^{\leftarrow}(O)}$ such that $p \in U \subseteq int_X(cl_X(U)) \cap f^{\leftarrow}(0) \subseteq A \cap f^{\leftarrow}(0)$, i.e. $\langle p_i \rangle_{i \in I} \in \prod_{i \in I} U_i \subseteq \prod_{i \in I} (int_{X_i}(cl_{X_i}(U_i)) \cap f_i^{\leftarrow}(O_i)) \subseteq \prod_{i \in I} (A_i \cap f_i^{\leftarrow}(O_i))$. So, O_k is an open neighbourhood of $f_k(x)$ in Y_k and U_k is an open set of $f_k^{\leftarrow}(O_k)$ such that $x \in U_k \subseteq int_{X_k}(cl_{X_k}(U_k)) \cap f_k^{\leftarrow}(O_k) \subseteq V \cap f_k^{\leftarrow}(O_k)$. Thus, by 2.2, the function $f_k \in C(X_k, Y_k)$ is semiregular.

($\Leftarrow$) If each f_i is semiregular, by 2.7, we have that $f_i = f_i(s)$ and so, by 3.5, $f = \prod_{i \in I} f_i = \prod_{i \in I} f_i(s) = \left(\prod_{i \in I} f_i \right)(s) = f(s)$ and, by 2.7, f is semiregular.

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